SEQUENTIAL THIRD ORDER ROTATABLE DESIGNS UP TO ELEVEN FACTORS

By P. J. Thaker and M. N. Das

I.A.R.S., New Delhi

(Received on June 27, 1961)

1. Introduction

In statistical designs of experiments in the field of industry (especially in the chemical industry), numerous problems involve the fitting of a response surface, in which the response depends on one or more controllable variables or factors. Box and Hunter [1957] introduced a new class of designs called rotatable designs which permit a response surface to be fitted easily and provide spherical information contours. A third order rotatable design aids the fitting of such a cubic surface.

Let k be the number of independent variables or factors and let $(x_{1u}, x_{2u}, \ldots, x_{ku})$ be the levels of these factors for the u-th of the N experimental points in the k-dimensional factor space $(u = 1, 2, \ldots, N)$. Let η_u be the expectation of the response at the u-th experimental point. It is assumed that the response surface may be approximated by a third degree surface within the range of interest, that is,

$$y_u = \beta_0 x_{0u} + \sum_{i=1}^{k} \beta_i x_{iu} + \sum_{i \leqslant j=1}^{k} \beta_{ij} x_{iu} x_{ju} + \sum_{i \leqslant j \leqslant l=1}^{k} \beta_{ijl} x_{iu} x_{ju} x_{lu} \tag{1.1}$$

where x_{0u} is a dummy variate and the measurements of the k factors have been coded such that $\sum_{u=1}^{N} x_{iu} = 0$, $(i = 1, 2, \ldots, k)$. A set of N points is said to form a rotatable design of third order in k factors if the following relations hold;

(A) $\Sigma x_{1u}^2 = \Sigma x_{2u}^2 = \ldots = \Sigma x_{ku}^2 = \lambda_2 N,$

(B) $\Sigma x_{1u}^4 = \Sigma x_{2u}^4 = \ldots = \Sigma x_{ku}^4 = 3 \Sigma x_{iu}^2 x_{ju}^2 = 3\lambda_4 N,$

$(i \neq j = 1, 2, \ldots, k),$

(C) $(C_1)\ \Sigma x_{1u}^6 = \Sigma x_{2u}^6 = \ldots = \Sigma x_{ku}^6 = 5 \Sigma x_{iu}^4 x_{ju}^2,$

$(i \neq j = 1, 2, \ldots, k),$

$(C_2)\ \Sigma x_{iu}^4 x_{ju}^2 = 3 \Sigma x_{iu}^2 x_{ju}^2 x_{lu}^2 = 3\lambda_6 N,$

$(i \neq j \neq l = 1, 2, \ldots, k)$

$$\text{(D)} \quad (D_1)\ \frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2},$$

$$(D_2)\ \frac{\lambda_6\lambda_2}{\lambda_4^2} > \frac{(k+2)}{(k+4)}, \qquad (1.2)$$

and all other sums of powers and products upto and including order six are zero. The summations are over all design points, $u = 1, 2, \ldots, N$. The relations (D_1) and (D_2) ensure the estimation of the terms involved in the third degree equation (1.1) and are also independent of the scale. By convention, the scale of the design is normally adjusted so that $\lambda_2 = 1$. This adjustment is a convenience and not essential. In this paper, scaling has been performed only for a few designs where the tables are prepared to give the values of the parameters λ_4 and λ_6 for different values of N, the total number of points. The purpose to prepare such tables is to provide a basis for the selection of a more useful design in the sense that design with larger differences in the relations (D_1) and (D_2) are better, as indicated by Gardiner *et al.* [1959] that if the relation (D_2) is very close to an equality then the linear and cubic coefficients are very poorly estimated.

2. Orthogonal Blocking in Rotatable Designs

Suppose a third order rotatable design is split into two sets of points hereafter called blocks such that each block is a complete second order rotatable design. An experimenter may try one of these second order designs (preferably one with smaller number of points) and approximate the response function, using a second order surface. If the second order surface is found adequate as the representation of the unknown function by noting evidence of goodness of fit, the experimenter then may stop at this stage. If the second order surface is observed inadequate, the experimental design is then usually augmented by the second block to permit the estimation of the coefficients in the third order surface (1.1) with the help of all the points in two blocks. Thus the experimental programme progresses sequentially, the first block providing the information for the next step.

Let

n_1 = number of noncentral points in the first block,

n_{10} = number of central points in the first block,

n_2 = number of noncentral points in the second block,

n_{20} = number of central points in the second block.

Then the condition for orthogonal blocking, as given by Gardiner *et al.* [1959], is

$$\frac{\sum_{u} {}_1\, x_{iu}^2}{\sum_{u} {}_2\, x_{iu}^2} = \frac{(n_1 + n_{10})}{(n_2 + n_{20})} \tag{2.1}$$

where Σ_1 and Σ_2 in the numerator and denominator denote the summation for the values of u in the first block and second block respectively.

3. Generation of Points in k-Dimensions

The method, as explained by Bose and Draper [1959] for generation of point sets in k-dimensions, is used here and is discussed below briefly. Let $(x_1, x_2, \ldots, x_k)$ be a point set in k-dimensions and let P_k be the symmetric group of order k, that is, the group of all permutations of k elements. By operating upon $(x_1, x_2, \ldots, x_k)$ with the elements of P_k, we can obtain $k!$ point sets provided all the elements are non-zero and distinct. The point set $(x_1, x_2, \ldots, x_k)$ will supply 2^k points given by $(\pm x_1, \pm x_2, \ldots, \pm x_k)$ with the ith factor at two levels $+ x_i$ and $- x_i$ $(i = 1, 2, \ldots, k)$, and therefore the total number of design points generated through the initial point set $(x_1, x_2, \ldots, x_k)$ is $k!\, 2^k$.

4. Sequential Third Order Rotatable Design in Three Factors

Consider the following two blocks, each constituting a second order rotatable design;

Block (i)

(i) $C_2^3 \times 2^2 = 12$ points of the point set $(\sqrt{2}a, \sqrt{2}a, 0)$,

(ii) $C_1^3 \times 2 = 6$ points of the point set $(2^{\frac{3}{4}} a, 0, 0)$,

plus n_{10}, the requisite number of centre points. Total number of points in block (i) $= 18 + n_{10}$.

Block (ii)

(i) 16 points of the doubly replicated point set (a, a, a),

(ii) $C_1^3 \times 2 = 6$ points of the point set $(\sqrt{3 \cdot 818662}\, a, 0, 0)$,

(iii) $C_1^3 \times 2 = 6$ points of the point set $(\sqrt{1 \cdot 190709}\, a, 0, 0)$,

plus n_{20}, the requisite number of centre points. Total number of points in block (ii) $= 28 + n_{20}$.

From Equation (2.1), we get

$$n_{10} = 5 \cdot 3060 + 0 \cdot 8324 n_{20} \quad (4.1)$$

for determining n_{10} and n_{20} as required for orthogonal blocking. Given below in the table are some possible values of n_{10}, n_{20}, N, λ_4, λ_6 and a^2 so that $\sum_{u=1}^{N} x_{iu}^2 = N$. The total number of non-central points in the design is $18 + 28 = 46$.

n_{10}	n_{20}	N	λ_4	λ_6	$\frac{5}{7}\lambda_4^2$	a^2
5	0	51	·7180	·3840	·3682	1·0697296
6	1	53	·7462	·4147	·3977	1·1116798
7	2	55	·7743	·4466	·4283	1·1536299
8	3	57	·8025	·4797	·4600	1·1955801
9	4	59	·8306	·5140	·4928	1·2375303

5. Sequential Third Order Rotatable Designs in Four Factors

Gardiner *et al.* [1959] obtained one sequential third order rotatable design with four factors in 128 noncentral points. Recently, Draper [1960] constructed a sequential third order rotatable design with four factors in 96 noncentral points. With $k = 4$, an infinite series of third order rotatable designs has been obtained. Firstly we are presenting a design with 72 points, and then four designs of the infinite series with 112, 72, 120 and 88 points.

5.1. *Design with* 72 *Points*

A sequential third order rotatable design may be formed from the points of the following three point sets:—

(i) $C_2^4 \times 2^2 = 24$ points of the point set

$(2^{\frac{2}{3}} a,\ 2^{\frac{2}{3}} a,\ 0,\ 0)$, (known as "truncated cube (2)"),

(ii) 32 points of doubly replicated point set (a, a, a, a),

(iii) $2\ (C_1^4 \times 2) = 16$ points of doubly replicated point set[3] $(2a, 0, 0, 0)$.

By double replication of a point set, we mean each point of the point set has to be taken twice. For sequential programming, the first block consists of the points of point set (1), *plus* n_{10}, the requisite number of centre points; while the second block consists of the points of point sets (ii), (iii), *plus* n_{20}, the requisite number of centre points. The total number of points in the first block and second block are $24 + n_{10}$ and $48 + n_{20}$ respectively. The equation (2.1) gives

$$n_{10} = 6.2381 + 0.6299\, n_{20}$$

while the values of n_{10}, n_{20}, N, λ_4, and λ_6 are given below together with the values of a^2 such that $\sum_{u=1}^{N} x_{iu}^2 = N$.

n_{10}	n_{20}	N	λ_4	λ_6	$\frac{6}{8}\lambda_4^2$	a^2
6	0	78	·7314	·4065	·4012	0·9969567
7	1	80	·7502	·4276	·4220	1·0225190
8	3	83	·7783	·4603	·4543	1·0608691
10	6	88	·8252	·5174	·5107	1·1247716

5.2. *Designs with* 112, 72, 120 *and* 88 *Points*

With the object of obtaining a third order rotatable design in four factors starting with the point set (a, a, a, b), known as "truncated cube (1)", it has been found that the following point sets form an infinite series of third order rotatable designs.

(i) $C_1^4 \times 2^4 = 64$ points of the point set (a, a, a, b),

(ii) $C_2^4 \times 2^2 = 24$ points of the point set $(c, c, 0, 0)$,

(iii) 8 points of the point set $(d, 0, 0, 0)$,

(iv) 8 points of the point set $(e, 0, 0, 0)$,

(v) 8 points of the point set $(f, 0, 0, 0)$,

(vi) 8 points of the point set $(g, 0, 0, 0)$.

From the relations (B), (C_1), (C_2) of (1.2), we get the following three equations,

$$u^2 + v^2 + s^2 + t^2 = -8p^2 + 48p + 24; \qquad (5.2.1)$$

$$8q^2 - 2(u^3 + v^3 + s^3 + t^3) = 16p^3 - 80p^2 - 80p - 112, \tag{5.2.2}$$

$$q^3 = -4p^2 + 32p + 4, \tag{5.2.3}$$

where $b^2 = pa^2$, $c^2 = qa^2$, $d^2 = ua^2$, $e^2 = va^2$, $f^2 = sa^2$, and $g^2 = ta^2$. By substituting the value of q^3 from (5.2.3) in (5.2.2), we get two equations in terms of p, u, v, s, t;

$$u^2 + v^2 + s^2 + t^2 = -8p^2 + 48p + 24, \tag{5.2.4}$$

$$u^3 + v^3 + s^3 + t^3 = -8p^3 + 24p^2 + 168p + 72. \tag{5.2.5}$$

By assigning a specified positive value to p, the value of q can be obtained from (5.2.3) while the values of u, v, s and t are to be assessed from (5.2.4) and (5.2.5). From (5.2.3), it is evident that the positive solution for q exists for $p \leqslant 4 + \sqrt{17}$. Similarly, the equations (5.2.4) and (5.2.5) give the limits of p for which the right-hand side of (5.2.4), (5.2.5) will attain positive number. This limit is $p \leqslant 3 + \sqrt{12}$. Thus, to bring positive solutions for q, u, v, s, t, the value of p must lie in the interval $(0, 3 + \sqrt{12})$.

The purpose of taking last four point sets (iii), (iv), (v), (vi) is to get positive solutions for u, v, s and t. Take minimum number of such point sets for which the positive solutions of u, v, s, t can be sought. For instance, let $p = 0$ then the equations (5.2.1), (5.2.2), (5.2.3) reduce to

$$q^3 = 4,$$

$$u^2 + v^2 + s^2 + t^2 = 24,$$

$$u^3 + v^3 + s^3 + t^3 = 72.$$

It can be seen that by taking only two point sets $(d, 0, 0, 0)$ and $(e, 0, 0, 0)$, the positive solutions for u and v cannot be obtained since the condition, according to Das [1961], that is

$$\frac{A^3}{2} \leqslant B^2 \leqslant A^3$$

where $u^2 + v^2 = A$, $u^3 + v^3 = B$, and A and B are positive numbers, is not satisfied. But, by adding the points of the point set $(f, 0, 0, 0)$ it is possible to get positive solutions for u, v, s. Though there exists infinite solutions for u, v, s, we will present only one solution for u, v, s since our aim is not to exhaust all possible solutions but simply to illustrate how the solutions can be assessed. The solutions for $q^3 = 4$, $v = 2$, are

$$v = 3 \cdot 4324565$$

$$s = 2 \cdot 8667477$$

and obviously $t = 0$.

The resultant design, with 112 non-central points, can be fitted into sequential programming by taking the points of point set (ii), *plus* n_{10}, the requisite number of centre points in the first block. The second block consists of points derivable from the point sets (i), (iii), (iv), (v) *plus* n_{20}, the requisite number of centre points. The equation (2.1), which expresses the orthogonal blocking, gives

$$n_{10} = 1 \cdot 9494 + 0 \cdot 2948\, n_{20}.$$

The values of n_{10}, n_{20}, N, λ_4, λ_6 and a^2 are given below as earlier:—

n_{10}	n_{20}	N	λ_4	λ_6	$\frac{6}{8}\lambda_4^2$	a^2
2	0	114	·6856	·3553	·3525	1·362866
3	4	119	·7157	·3871	·3841	1·422641

In the above design with 112 points, we may notice that the points of the point set $(a, a, a, 0)$ are doubly replicated. However, it is possible to derive a design with single replication of the point set $(a, a, a, 0)$ giving 32 points, in conjuction with point sets (ii), (iii), and (iv). The solutions for q, u, v, in this case, are

$$q^3 = 2,$$

$$u = 3 \cdot 247411,$$

$$v = 1 \cdot 205952,$$

and obviously $s = 0$, $t = 0$. The resultant design contains 72 non-central points.

Another design can be obtained in 120 noncentral points by putting $p = 1$ in (5.2.1), (5.2.2), and (5.2.3). The resultant three equations are

$$q^3 = 32,$$

$$u^2 + v^2 + s^2 + t^2 = 64,$$

$$u^3 + v^3 + s^3 + t^3 = 256.$$

The solutions for u, v, s, and t are $u = v = s = t = 4$.

The fourth design in 88 noncentral points can be obtained by putting $p = 3 + \sqrt{12}$ in (5.2.1), (5.2.2), and (5.2.3). The resultant three equations are

$$q^3 = 16\,(1 + \sqrt{3}),$$

$$u^2 + v^2 + s^2 + t^2 = 0,$$

$$u^3 + v^3 + s^3 + t^3 = 0,$$

which ultimately give $u = 0,\ v = 0,\ s = 0,\ t = 0$. The last two designs can be fitted into sequential plan by taking the points of point set (ii), *plus* n_{10}, the requisite number of centre points in the first block and the points of the remaining point sets, plus n_{20}, the requisite number of centre points in the second block. The values of n_{10}, n_{20} are to be chosen so as to satisfy the equation (2.1) of orthogonal blocking, while the scaling factor a, is chosen so that $\sum_{u=1}^{N} x_{iu}^2 = N$.

6. Sequential Third Order Rotatable Design in Five Factors

A third order rotatable design in five factors may be performed sequentially in the following two blocks.

Block (i)

(i) 32 points of the point set (a, a, a, a, a);

(ii) 10 points of the point set $(2^{5/4}a, 0, 0, 0, 0)$,

plus n_{10}, the requisite number of centre points. Total number of points in block (i) $= 42 + n_{10}$.

Block (ii)

(i) $C_3^5 \times 2^3 = 80$ points of the point set $(\sqrt{2}a, \sqrt{2}a, \sqrt{2}a, 0, 0)$;

(ii) $C_2^5 \times 2^2 = 40$ points of the point set $(2^{\frac{2}{3}} a, 2^{\frac{2}{3}} a, 0, 0, 0)$,

(iii) 10 points of the point set $(\sqrt{5 \cdot 8693939}\, a, 0, 0, 0, 0)$,

(iv) 10 points of the point set $(\sqrt{0 \cdot 9225051}\, a, 0, 0, 0, 0)$,

plus n_{20}, the requisite number of centre points. Total number of points in block (ii) $= 140 + n_{20}$.

The resultant design contains 182 noncentral points while the values of n_{10}, n_{20} are to be chosen so as to satisfy the equation (2.1) of orthogonal blocking. Again, the scaling factor a is chosen so that $\sum x_{iu}^2 = N$.

7. Sequential Third Order Rotatable Design in Six Factors

A sequential third order rotatable design in six factors consists in performing experiments in the following two blocks.

Block (i)

(i) 64 points of the point set (a, a, a, a, a, a),

(ii) 12 points of the point set $(2^{3/2}a, 0, 0, 0, 0, 0,)$,

plus n_{10}, the requisite number of centre points. Total number of points in block (i) $= 76 + n_{10}$.

Block (ii)

(i) $C_3^6 \times 2^3 = 160$ points of the point set $(2^{2/3}a, 2^{2/3}a, 2^{2/3}a, 0, 0, 0)$,

(ii) 24 points of the doubly replicated point set $(\sqrt{2}a, 0, 0, 0, 0, 0)$,

plus n_{20}, the requisite number of centre points. Total number of points in block (ii) $= 184 + n_{20}$.

The resultant design contains 260 noncentral points and the values of n_{10}, n_{20} are to be determined from the equation (2.1) of orthogonal blocking. The value of a, the scaling factor, is chosen so that $\Sigma x_{iu}^2 = N$.

8. Use of Fractional Replicates

To reduce the size of the experiment, a suitable fractional factorial can replace the full factorial. Considerable savings were demonstrated by Box and Hunter [1957] in the case of second order rotatable designs by the use of fractional replication for $k > 4$. For a third order rotatable design, we must have at least seven factors to make use of fractional replication. When there are seven or more factors, we have to take that fraction of the total number of combinations which can be obtained through an identity group of interactions in which there is no interaction with less than seven factors. The half replicate of the point set $(a, a, a, \ldots, a)$ in k-factors is the half replicate of the 2^k factorial design with levels $+a$ and $-a$ for all the factors. We will use the usual fractional factorial notation, $\frac{1}{2}(2^k)$, to denote the half replicate of the point set $(a, a, a, \ldots, a)$.

9. Sequential Third Order Rotatable Design in Seven Factors

A sequential third order rotatable design in seven factors may be performed in two stages of experimentation by taking the following two blocks:—

Block (i)

(i) $\frac{1}{2}(2^7) = 64$ points of the half replicated point set (a, a, a, a, a, a, a),

(ii) 14 points of the point set $(\sqrt{7 \cdot 542072}a, 0, 0, 0, 0, 0, 0)$,

(iii) 14 points of the point set $(\sqrt{2 \cdot 667788}a, 0, 0, 0, 0, 0, 0)$,

plus n_{10}, the requisite number of centre points. Total number of points in block (i) $= 92 + n_{10}$.

Block (ii)

(i) $C_3^7 \times 2^3 = 280$ points of the point set $(\sqrt{2}a, \sqrt{2}a, \sqrt{2}a, 0, 0, 0, 0)$

plus n_{20}, the requisite number of centre points. Total number of points in block (ii) $= 280 + n_{20}$.

The resultant design contains 372 noncentral points and the values of n_{10}, n_{20} are to be calculated from equation (2.1) of orthogonal blocking. The scaling factor a has to be so chosen that $\Sigma x_{iu}^2 = N$.

10. Sequential Third Order Rotatable Designs in Seven and More Number of Factors

Of greater interest are those point sets which may form sequential third order rotatable designs in seven factors and may be extended to more than seven factors. In doing so, it was found that the following point sets constitute sequential third order rotatable designs up to eleven factors.

(i) $\frac{1}{2}(2^k) = 2^{k-1}$ points of the half replicated point set $(a, a, a, \ldots, a)$,

(ii) $\frac{1}{2}(2^k) = 2^{k-1}$ points of the half replicated point set $(b, b, b, \ldots, b)$,

(iii) $C_3^k \times 2^3$ points of the point set $(c, c, c, 0, 0, \ldots, 0)$, where the element '$c$' occurs thrice and zero occurs $(k-3)$ times,

(iv) $2k$ points of the point set $(d, 0, 0, \ldots, 0)$,

(v) $2k$ points of the point set $(e, 0, 0, \ldots, 0)$,

(vi) $2k$ points of the point set $(f, 0, 0, \ldots, 0)$,

(vii) $2k$ points of the point set $(2^{(k-1)/4} a, 0, 0, \ldots, 0)$.

For a design to be performed sequentially, the first block consists of a set of points of the point sets (i) and (vii) while the second block comprises of the point sets (ii), (iii), (iv), (v) and (vi).

The relations (B), (C_1), (C_2) of (1.2) give respectively the following equations:

$$d^4 + e^4 + f^4 = 2^{k-1} b^4 + 2(k-2)(7-k) c^4, \tag{10.1}$$

$$2^{k-1}(a^6 + b^6) + 2(2^{3(k-1)/2} a^6 + d^6 + e^6 + f^6) + C_2^{k-1} 2^3 c^6$$
$$= 5\{2^{k-1}(a^6 + b^6) + (k-2) 2^3 c^6\}, \tag{10.2}$$

$$c^6 = \frac{2^{k-3}(a^6 + b^6)}{(k-5)}. \tag{10.3}$$

Substituting the value of c^6 from (10.3) in (10.2) and after simplifying, we have

$$d^6 + e^6 + f^6 = 2^{k-1}\left[2(a^6 + b^6) + \frac{(a^6 + b^6)}{(k-5)} \times \left\{5(k-2) - \frac{(k-1)(k-2)}{2}\right\} - 2^{(k-1)/2}\right], \tag{10.4}$$

Let $b^2 = pa^2$, $c^2 = qa^2$, $d^2 = ua^2$, $e^2 = va^2$, and $f^2 = sa^2$, then the equations (10.1), (10.4), (10.3) reduce to

$$u^2 + v^2 + s^2 = 2^{k-1} p^2 + 2(k-2)(7-k) q^2, \tag{10.5}$$

$$u^3 + v^3 + s^3 = 2^{k-1}\left[2(1 + p^3) + \frac{(1 + p^3)}{(k-5)} \times \left\{5(k-2) - \frac{(k-1)(k-2)}{2}\right\} - 2^{(k-1)/2}\right]. \tag{10.6}$$

and

$$q^3 = \frac{2^{k-3}(1 + p^3)}{(k-5)}. \tag{10.7}$$

For a sequential third order rotatable design in k factors $(k \geqslant 7)$, the positive solutions for p, q, u, v and s are so chosen that the equations (10.5), (10.6), and (10.7) are satisfied. Again the scaling factor a, is so chosen that $\Sigma x_{iu}^2 = N$.

With $k = 7$, an infinite series of sequential third order rotatable designs has been obtained. Putting $k = 7$ in (10.5), (10.6), (10.7), we get

$$u^2 + v^2 + s^2 = 2^6 p^2, \tag{10.8}$$

$$u^3 + v^3 + s^3 = 2^6 \{7(1 + p^3) - 8\}, \tag{10.9}$$

and

$$q^3 = 8(1 + p^3). \tag{10.10}$$

It is to be noted that the right-hand side of equation (10.9) will attain positive number only when $p^3 \geqslant 1/7$. As mentioned earlier, the purpose of taking three point sets (iv), (v) and (vi) in the second block is to achieve positive solutions for u, v and s. We put $s = 0$, since it is possible to get positive solutions for u and v. Putting $s = 0$, the equations (10.8), (10.9) reduce to

$$u^2 + v^2 = 2^6 p^2 = A \text{ (say)}, \tag{10.11}$$

$$u^3 + v^3 = 2^6 \{7(1 + p^3) - 8\} = B \text{ (say)}, \tag{10.12}$$

where A and B are real positive numbers. According to Das [1961], the positive solutions for u, v are accessible if

$$\frac{A^3}{2} \leqslant B^2 \leqslant A^3$$

i.e.,

$$0 \leqslant (17p^6 - 14p^3 + 1)\, 2^{12} \leqslant 2^{12} (32p^6) \tag{10.13}$$

There will be infinite values of p satisfying the condition (10.13) and for each such value of p, we will get a rotatable arrangement. For instance, the values of p^3 which satisfy the left-hand equality of (10.13) are given by

$$p^3 = \frac{(7 \pm \sqrt{32})}{17}.$$

For

$$p^3 = \frac{(7 + \sqrt{32})}{17},$$

the solutions for u, v are

$$u = v = 5 \cdot 127048.$$

The resultant design contains 450 noncentral points with the values of p, q, u, v and s given by

$$p^3 = \frac{(7 + \sqrt{32})}{17},$$

$$q^3 = \frac{8\,(24 + \sqrt{32})}{17},$$

$u = v = 5{\cdot}127048$, and obviously $s = 0$. For $p^3 = (7 - \sqrt{32})/17$, we get negative value of B in (10.12) and thus the positive solutions for u and v cannot be obtained. It can be seen that the limits for p^3 are

$$\frac{1}{7} \leqslant p^3 \leqslant \frac{(7 + \sqrt{32})}{17}.$$

The value of p^3 which lie inside this limit will always furnish positive solutions for u and v.

It is now evident that the derivation of sequential third order rotatable designs can be sought by assigning the positive specified values to p, q, u, v and s so that the equations (10.5), (10.6), and (10.7) are satisfied. The table given below shows the values of p^3, q^3, u, v, and s for $k = 8, 9, 10, 11$. Also given are the total number of points contained in each block. The designs presented here are simply illustrative and are not exhaustive.

TABLE

The value of p^3, q^3, u, v and s for $k = 8, 9, 10, 11$

k	The number of experimental points in block (i)	The number of experimental points in block (ii)	N	p^3	q^3	u	v	s
8	$144 + n_{10}$	$608 + n_{20}$	$752 + n_{10} + n_{20}$	8	96	15·005–0688	5·937–7388	0
9	$274 + n_{10}$	$964 + n_{20}$	$1238 + n_{10} + n_{20}$	5	96	11·426–6036	5·561–9408	0
10	$532 + n_{10}$	$1512 + n_{20}$	$2044 + n_{10} + n_{20}$	12	$2^7 \times 2{\cdot}6$	19·137–1124	5·603–2760	0
11	$1046 + n_{10}$	$2410 + n_{20}$	$3456 + n_{10} + n_{20}$	27	$\frac{2^{10} \times 7}{3}$	24·182–8710	11·508–9848	18·8

The total number of points, N, can be reduced for $k = 11$ by taking ¼ replicates of the point sets (i) and (ii).

11. SUMMARY

Gardiner *et al.* [1959] studied in details the third order multifactorial rotatable designs for fitting a third order (*i.e.*, a cubic) surface and constructed sequential third order rotatable designs up to four factors. Draper [1960] constructed infinite series of third order rotatable designs in three dimensions and also obtained one third order rotatable design in 96 noncentral points in four dimensions. Recently, Das [1961] suggested a further method using a modified form of factorial design and obtained through this method sequential as well as non-sequential third order rotatable designs up to seven factors. In the present investigation, sequential third order rotatable designs upto eleven factors have been obtained. Most of these designs possess the desirable property of having small number of experimental points.

12. ACKNOWLEDGEMENT

We are thankful to Dr. G. R. Seth, Statistical Adviser, I.C.A.R., for providing necessary facilities and guidance for the present investigation.

13. REFERENCES

1. Bose, R. C. and Draper, Norman R. .. "Second order rotatable designs in three dimensions," *Ann. Math. Stat.*, 1959, **30**.
2. Box, G. E. P. and Hunter, J. S. "Multi-factor experimental designs for exploring response surfaces," *ibid.*, 1957, **28**.
3. Das, M. N. .. "Construction of rotatable designs from factorial designs," 1961.
4. Draper, Norman R. .. "Third order rotatable designs in three dimensions," *Ann. Math. Stat.*, 1960, **31**.
5. ——— .. "A third order rotatable design in four dimensions," *ibid.*, 1960, **31**.
6. Gardiner, D. A., Grandage, A. H. E. and Hader, R. J. "Third order rotatable designs for exploring response surfaces," *ibid.*, 1959, **30**.